# Different topologies for a herding model of opinion

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Understanding how opinions spread through a community or how consensus emerges in noisy environments can have a significant impact on our comprehension of social relations among individuals. In this work a model for the dynamics of opinion formation is introduced. The model is based on a nonlinear interaction between opinion vectors of agents plus a stochastic variable to account for the effect of noise in the way the agents communicate. The dynamics presented is able to generate rich dynamical patterns of interacting groups or clusters of agents with the same opinion without a leader or centralized control. Our results show that by increasing the intensity of noise, the system goes from consensus to a disordered state. Depending on the number of competing opinions and the details of the network of interactions, the system displays a first- or a second-order transition. We compare the behavior of different topologies of interactions: one-dimensional chains, and annealed and complex networks.

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## I. INTRODUCTION

An interesting application concerning the structure of social networks [1,2] is the modeling of the dynamics of opinion formation. Specific measurements that characterize the statistics behind the existence of different groups and affiliations within human populations justify modeling such an aspect of human behavior. In this field the idea is to find simple rules of interactions behind the nodes or *agents*, each of which carries its own changing color or *opinion*, trying to reproduce the emergence of complex patterns observed in reality.

Such opinions can be defined by a finite number of integers as in the model proposed by Sznajd-Weron and Sznajd [3] or can even be represented by *real* numbers, having a rich spectrum and opening the possibility for having as many opinions as agents; as in the model proposed by Deffuant *et al.* [4]. In both cases the proposed dynamics has a natural absorbing state or *consensus*, in which all the agents share the same opinion. Other models such as the one by Hegselmann and Krause [5], the voter model [6], Galam's majority rule [7], and Axelrod's model [8] are reviewed in Ref. [9]. As pointed out there, very few and nonconclusive results exist for the consensus models on complex networks.

In this work, we present a general model, where opinions are represented by vectors with real components and the agents interact with a nonlinear rule. We propose, for the abstract space of human opinions, a dynamical rule where each agent has an opinion vector that is fixed in modulus. Every time step an agent interacts with its neighbors and assumes a new value for its opinion vector that is a function of the average direction of its neighboring agents plus an added noisy term. The resulting behavior presents two important characteristics: (i) although the model allows a continuous change from one opinion to some other, the interaction favors extremely decided states over undecided states, and (ii) the system ubiquitously evolves into coordination and grouping without the need of leaders or centralized control. The fact that the modulus of the opinion vector is constant describes the strength of an opinion about a specific topic at the expense of the other beliefs. According to our model, undecided agents (i.e., those that do not have a strong belief in one particular opinion) cannot affect the ones with a strong opinion. This type of interaction is somewhat different from the one used in models of opinion formation, which usually consists in weighted averages. Similar rules have been proposed, for example, in Ref. [10] to explain how very large populations are able to converge to the use of a particular word or grammatical construction without global coordination.

We study the transition to consensus as a function of noise. We find that different types of transitions with or without hysteresis are observed depending on the dimension of the opinion vector. Additionally, we observe that the transition is controlled by the interaction dynamics and is independent of the correlations of the network topology.

We start in Sec. II by describing in detail the proposed model for interactions between opinions. In Sec. III, we present the results of the transition to consensus as a function of the noise for one-dimensional (1D) chains, and annealed and complex networks. Conclusions are given in Sec. IV.

## II. OPINION MODEL

The system comprises a fixed number of $N$ agents. Every agent $i$ is characterized by its own opinion vector $a_n^{(i)}$ of $n=1,\ldots,O$ opinions. Each element of this vector corresponds to a different opinion about the same topic. Negative values of the elements are not allowed. The sum of the vector elements of an agent is 1: $\Sigma_{n=1}^{O} a_n^{(i)}=1$. For instance, agent $j$ favors communism with 20% and capitalism with 80% (given $O=2$): $a_1^{(j)}=0.2$, $a_2^{(j)}=0.8$. Each time step every agent actualizes its opinion vector by comparing its values to the ones of its $k_i$ nearest neighbors. These other agents are cho-

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sen by the topology of the graph, and the agent updates its opinion vector due to the following rule:

$$\hat{a}_n^{(i)}(t) = \sum_{l=1}^{k_i} a_n^{(i)}(t) a_n^{(l)}(t) + k_i g(t), \tag{1}$$

where $g(t)$ presents a stochastic variable, distributed uniformly in the interval $[0, \eta]$. With this exclusively positive noise we are assured that $a_n^{(i)}(t) \geqslant 0$. This stochasticity can be interpreted to be due to misunderstandings among the agents, the spread of wrong information, or other perturbing actions.

The interaction term in this model is of second order. Thus, in a noiseless environment, the agents tend to have the same stronger opinion. The factor $k_i$ avoids that agents with more connections feel less noise. In order to guarantee that the sum of opinions is equal to 1, the vector is normalized afterward, similarly to the model presented in Ref. [11],

$$a_n^{(i)}(t+1) = \frac{\hat{a}_n^{(i)}(t)}{\sum_{m=1}^{O} \hat{a}_m^{(i)}(t)}. \tag{2}$$

In order to elucidate the principal properties of the update rule given by Eqs. (1) and (2), we examine in detail the noiseless interactions between three types of agents with different characteristic values of a two-dimensional opinion vector ($O=2$): namely, $a_1=\{0.8, 0.2\}$, $a_2=\{0.5, 0.5\}$, and $a_3=\{0.2, 0.8\}$. First, an interaction between an agent having $a_1$ with another having the same $a_1$ results in {0.94, 0.06}—interactions between agents with the same dominant opinion strengthen their belief in this opinion. $a_1$ with $a_2$ yields {0.8, 0.2}—interactions with "undecided" agents are ineffective in the sense that agents without dominant opinion are not able to convince another agent. On the other hand, this interaction will have a substantial effect on the undecided agent; i.e., undecided agents are convinced easily. The interaction between $a_1$ and $a_3$ results in {0.5, 0.5}—interactions of agents with opposite opinions lead them to become less decided.

At the beginning of a simulation the opinion vectors are initialized either randomly or by consensus: for random initialization we randomly select for each opinion component of each agent a number between 0 and 1. The opinion vectors are normalized afterwards according to Eq. (2). The other way to initialize the system (consensus) is by setting to 1 the first element of each vector and fill the rest with 0's.

The main parameter of this model is given by the maximal noise $\eta$ which we will call from now on the control parameter. Its role corresponds to the one of a temperature in physical systems. In a social system, the noise represents any internal or external interference in the communication among the agents. Other free parameters of the system are given by the number of agents, $N$, the number of opinions, $O$, and the number of agents, $k_i$, to interact with per time step. The last parameter can be different for distinct agents depending on the topology of the actual network.

A simple mean-field solution of the model without noise can be derived. Suppose a state where all agents have the same values in their opinion vectors. Thus the index of the agents can be suppressed, $a_n^{(i)}(t)=a_n(t)$ and $a_n^{(i)}(t+1)=a_n(t+1)$, and Eqs. (1) and (2) can be summarized. In the case of two opinions the equations correspond to the map,

$$a_1(t+1) = \frac{a_1^2(t)}{a_1^2(t)+a_2^2(t)}, \quad a_2(t+1) = \frac{a_2^2(t)}{a_1^2(t)+a_2^2(t)}. \tag{3}$$

The fixed points of these equations are $(a_1, a_2) \in \{(0,1),(1,0),(0.5,0.5)\}$ where the first two are stable and the last one is unstable. The solutions for $O$ opinions are in $\{(1,0,0,\ldots),(0,1,0,0,\ldots),\ldots,(0,0,\ldots,0,1)\}$ with all $a_n$ stable. All other solutions have at least one unstable element of the opinion vector, and thus the unstable element influences the other ones until an absorbing state with one opinion totally dominant is reached.

## III. RESULTS

### A. Annealed interactions

First, we present simulations of the model without fixed topology. Each time step, a simulation runs over all agents. For each of them and at each time step, two new random partners are chosen to interact. We chose an interaction with two other agents ($k_i=k=2$) in order to facilitate the comparison of this case with the one of a one-dimensional chain which will be explained in the following section. The annealed approach avoids long-term behavior, and the distribution of opinions reaches the stationary state fast. Because the interacting units are a sampling of the whole system, it is expected that this annealed approximation should behave similar to a mean field.

The results reveal that the system can reach two different absorbing states. At small values of the control parameter (maximum noise $\eta$), one opinion completely dominates the system, $o_{max}$. For a noise $\eta$ larger than a certain value, each opinion remains with the same frequency $1/O$. The order parameter $D$ is the frequency of agents which have an opinion vector with the same dominant opinion, being itself dominant in the system. More precisely, for each agent we search its strongest opinion and then count, for each opinion, the number $n$ of agents with this opinion as their dominant one. The largest value $n_D$, and so the most dominant one of the system, determines $D=n_D/N$. $\langle D\rangle$ means that we average $D$ over many time steps. This order parameter is normalized, so that it is unity if all agents have the same dominant opinion, a state we call the consensus state. The value $1/O$ corresponds to a uniform distribution of opinions. A transition occurs between consensus and uniform distributions, when $\langle D\rangle$ goes from 1 to 1/2 in the case of two opinions [Fig. 1(a)].

The transition becomes more abrupt for larger population sizes. A transition point characteristic for the jump from the consensus to the uniform states is located at $\eta_c \approx 0.5$, increasing with the population size. This transition seems to be a phase transition of second order. We carried out finite-size scaling in order to obtain the critical exponents [Fig. 1(b)]. Near the critical point the curves coincide using the scaling relations,

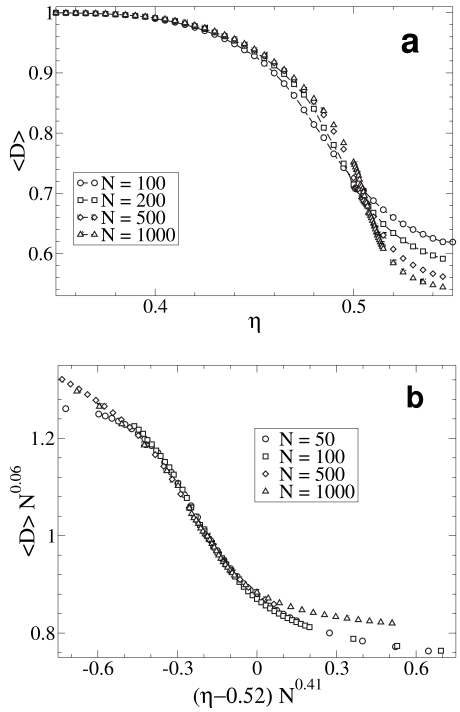


FIG. 1. Finite-size scaling of the transition in a system with a two-dimensional opinion vector with random interactions. The figure compares the frequency of agents with the same dominant opinion versus the noise $\eta$ for different population sizes $N$. (a) Original data. (b) Finite-size scaling. Each point corresponds to an average over 10–20 runs with different random seeds.

$$\langle D\rangle N^{-\beta/\nu} = (\eta - \eta_c)N^{1/\nu}, \tag{4}$$

with $\nu \approx 2.4 \pm 0.1$, $\beta = 0.15 \pm 0.05$, and the critical noise $\eta_c = 0.52$.

Figure 2 shows that in the case of annealed interactions the transition becomes of first order for simulations with an opinion vector of more than two opinion elements. The fluctuations do not increase at the transition point. Now, the transition from the consensus to the uniform state depends on the initialization and is much more abrupt. If the initialization is random, the system jumps to the consensus state at lower values of $\eta$ than in the case of an initialization with consensus in one opinion. A transition with a typical hysteresis occurs at lower values of $\eta$ if we increase the dimension of the opinion vector.

Note that $D$ gives us the fraction of agents with dominant opinion ($o_{max}$) but does not contain information about $a_{o_{max}}$, the magnitude of the component associated with $o_{max}$. In Fig. 2(d) we plot $a_{o_{max}}$ vs $\eta$ for the same simulations presented in Fig. 2(c). $a_{o_{max}}$ is larger for lower values of $\eta$, and below a certain $\eta_c$ consensus is observed for both kinds of initializations, *only* when a large value of $a_{o_{max}}$ is reached. This is a nice feature of our model: *consensus and resolution emerge together in the system*. That is, the agents can only make up their minds for a preferred opinion when consensus is achieved through the entire system.

### B. One-dimensional topology

If we put the agents on a one-dimensional lattice with periodic boundary conditions, or, in other words, a chain, the results become different. First, we concentrate on the case of ten opinions and no noise ($\eta = 0$): The system is now highly dependent on the initial state. A random initialization of the opinion vectors leads to the situation depicted in Fig. 3(a). The same amount of each opinion seems present in the system during the evolution. The system organizes itself by rearranging its opinion vectors to form local clusters of different sizes. In one cluster the same opinion dominates for all

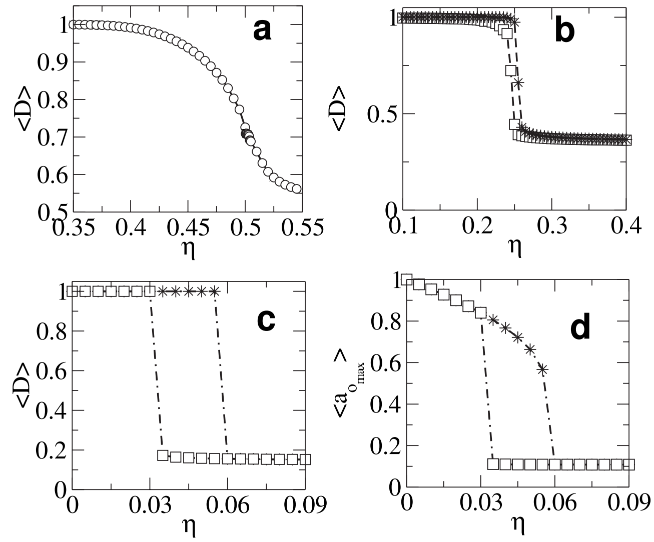


FIG. 2. The transition of the system for different numbers of opinions, $O$, and different initial conditions. The population size is 500. (a) The outcome of a system of two opinions (circles) is independent of its initialization. (b) In the case of three opinions (dashed line), the curves present a hysteresis and the results are different if the field is initialized randomly (squares) or with the system being already in the consensus state (stars). (c) A system with an opinion vector containing ten opinions also exhibits hysteresis (dash-dotted line). (d) Average value of the dominant opinion $a_{o_{max}}$ vs $\eta$ for the same simulation as in (c).

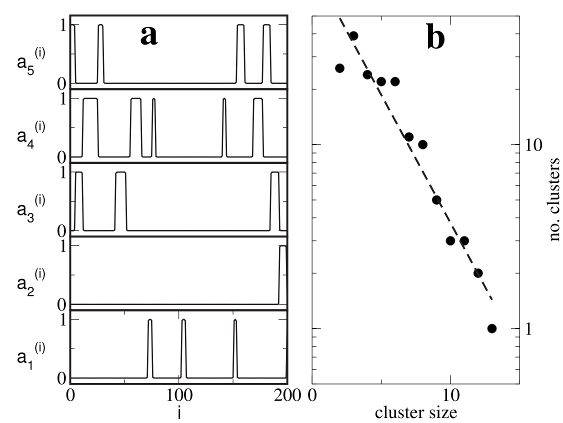


FIG. 3. Ten opinions on a one-dimensional chain. These results illustrate simulations without noise ($\eta = 0$) and a system of 1000 agents. (a) The figure zooms in on the first 200 agents of the population, where each line corresponds to a different element of the opinion vectors. Only the first five opinions are displayed. The agents form local clusters of different dominant opinions. (b) The cluster sizes distribute following an exponential decay.

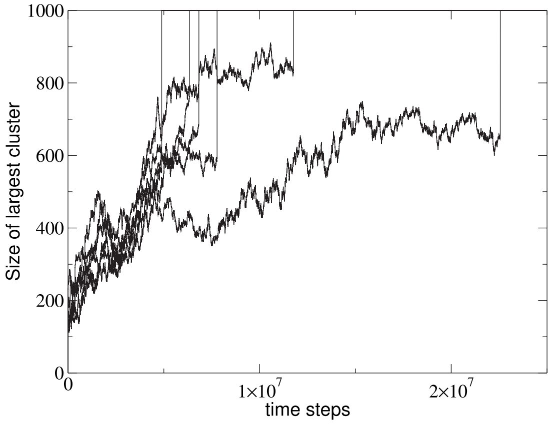


FIG. 4. The size of the largest cluster increases in time until it reaches the population size $N=1000$. Here, we see the results of different random initializations of the system on a chain. The number of opinions is 10, and the noise $\eta=0.2$.

agents. Each agent has a well-pronounced dominance of an opinion (its value being nearly 1), and the interfaces between clusters of different dominant opinions are very sharp. These clusters develop fast after the beginning of the simulation. The distribution of cluster sizes follows the exponential decay of a Poisson distribution [Fig. 3(b)]. The results with $\eta=0$ are qualitatively the same for different numbers of opinions, $O$.

Noise ($\eta>0$) leads to a slow increase of one of the ten opinions with time. The dominant opinion absorbs more and more of the losing opinions. Figure 4 illustrates how the largest cluster of the system temporally evolves for $\eta=0.2$. As also can be recognized in this figure, the time to reach consensus can be really long, even in a small system of 1000 agents.

With nonzero but small noise, the information propagates slowly through the sample. Because of that, the time to reach the absorbing state is much larger than in the case of random interactions. Next we consider a system consisting of 1000 agents, which have opinion vectors of two dimensions. The normalization of Eq. (2) allows us to focus only on the temporal behavior of one of each agent's opinion without loss of information. Figure 5 exhibits this time behavior for noises $\eta=0.05, 0.2, 0.35, 0.45$ during the first 100 000 time steps. Each agent's first opinion is depicted by a color (gray tone) which corresponds to its value and evolves beginning at the bottom. At low noise values stable clusters seem to form. The size of the clusters becomes smaller with decreasing $\eta$. Nevertheless, these clusters are not stable, and the system reaches the consensus state after a finite time. For $\eta=0.05$ and $\eta=0.2$ the size of clusters with the second dominant opinion is larger, indicating that at the end this opinion will control the system. The larger the size of a cluster, the longer it takes to break it. At values of $\eta$ larger than $\eta=0.3$, strong fluctuations control the system and consensus begins to become unstable. For values around $\eta=0.3$ one opinion still dominates and clusters appear and disappear. At larger $\eta$ the opinions have values around 0.5 for all agents which do not fluctuate much.

It is interesting to calculate the number of time steps the system needs to reach its final state. In a system of two opinions we carried out various simulations with the same value of noise, $\eta$, and a population size of $N=100$. Each simulation begins with an initialization of randomly distributed opinions but a different random seed. Figure 6 shows the distribution of times needed to reach the consensus state for a system. The distribution decreases exponentially. The distribution becomes broader with decreasing values of $\eta$, where $\eta=0$ should correspond to a flat distribution.

As in the case of random interactions, a transition occurs from the consensus state to the one of a uniform distribution of the opinions. Figure 7(a) illustrates the variation of $\langle D\rangle$ with $\eta$ for different system sizes. As shown in Fig. 7(b), by

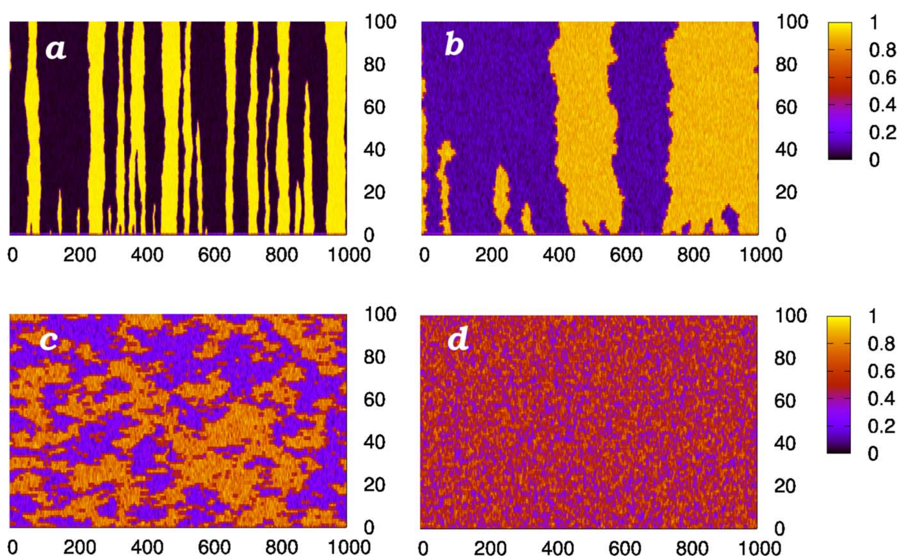


FIG. 5. (Color online) Graphical illustration of the temporal behavior of the system on a chain. The color (gray tone) corresponds to the value of the first of the two opinions of the system. The simulations run over 100 000 time steps, drawing each 1000 iterations a new point on the vertical axis beginning at the bottom. The horizontal axis depicts the location of each agent on the chain, altogether consisting of 1000 agents. The noise $\eta$ is 0.05 in (a), 0.2 in (b), 0.35 in (c), and 0.45 in (d).

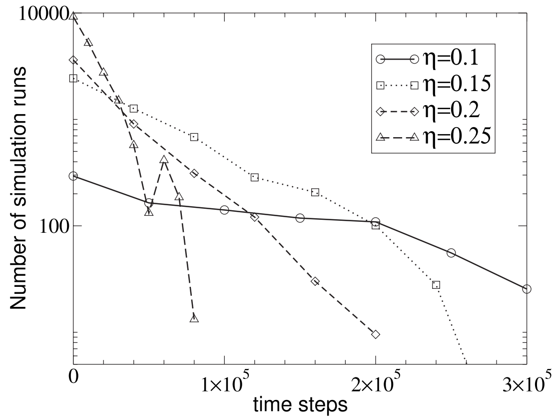


FIG. 6. Histogram of the time steps needed to reach the consensus state. Each curve corresponds to simulations with the same parameters: 100 agents, 2 opinions, random initialization. For each value of the noise parameter we carried out 200 000 runs with different random seeds.

performing a finite-size scaling analysis through Eq. (4), the collapse of all curves is obtained when we use the critical exponents $\nu=2$ and $\beta=2$.

### C. Complex networks

In this section we compare the behavior of the opinion model if the agents interact with their $k$ nearest neighbors on different network topologies. We study two different kinds of

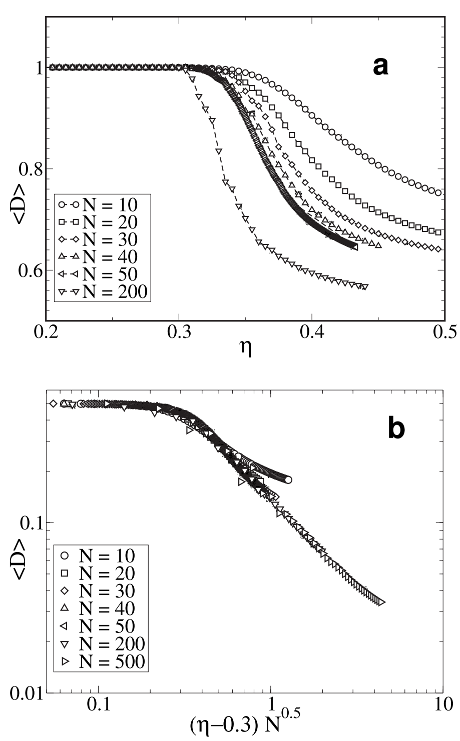


FIG. 7. (a) The fraction of agents with the same dominant opinion versus $\eta$ are compared for different population sizes $N$ in a system of two opinions, where only the nearest neighbors on a chain interact. (b) Finite-size scaling of the transition.

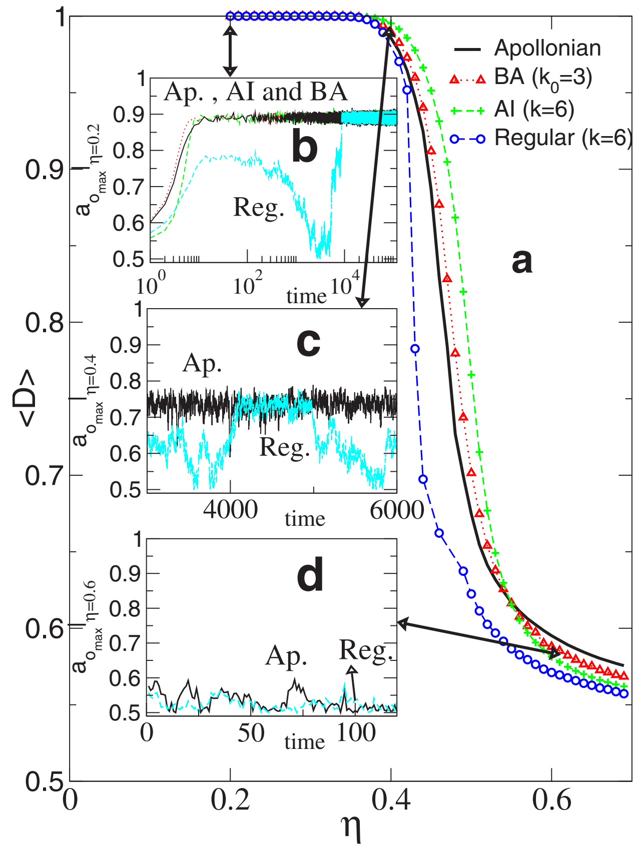


FIG. 8. (Color online) (a) Influence of the topology of networks on the transition to consensus ($D=1$) as a function of noise ($\eta$). The transition on two different scale-free networks, the Apollonian (solid line) and the Barabási-Albert network (BA, triangles), is similar to the one observed for annealed interactions (AI, plus signs), and differs from the transition on a regular lattice (circles). In the three insets we plot the value of the dominant opinion, $a_{o_{\max}}$ vs time. (b) Comparison of the behavior of $a_{o_{\max}}(t)$ on the four networks: Apollonian (solid line), Barabási-Albert (BA, dotted line), annealed (AI, dashed line) and regular (long dashed line) for a fixed noise ($\eta=0.2$). One observes that for this noise, which is below the critical noise, in the regular network the emergence of consensus takes longer than in scale-free and annealed interactions, which have similar behavior (three upper curves). (c) Near below the transition, for $\eta=0.4$, we compare the response of the regular and the Apollonian network. It is observed that for the former there is an intermittency among consensus $D=1$ and $a_{o_{\max}}=0.79$ and not consensus $D=0.5$ and $a_{o_{\max}}=0.5$. This behavior is not observed in the complex networks. (d) Above the transition ($\eta=0.6$) the consensus is broken and the dynamics of the opinion $a_{o_{\max}}$ vs time behaves similarly in regular and complex networks. All simulation runs are with systems of 124 agents.

scale-free networks: i.e., networks with a power law degree distribution $k^{-\alpha}$. Those are the Barabási-Albert (AB) network [1] and the Apollonian network [12]. The networks have considerable topological differences, which can be expressed in terms of their clustering coefficient $C$. This coefficient is the average probability that the neighbors of a node are connected among them. The BA network has a clustering coefficient $C$, which depends on the network size as $N^{-1}$. It is independent of the degree of the nodes. In contrast, the Apollonian network has hierarchical structure with $C$ depending on the degree of the node as a power law of the degree and its average value is high ($C \approx 0.8$) and independent of the network size $N$. Both types of scale-free networks, with and without hierarchical structure, have shown to be good models for rather different kinds of social interaction networks, from social collaboration networks [13] to networks of sexual contacts [14].

Further, we show that despite the structural differences of these networks, the formation of consensus depends mainly on the noise and is independent of the specific topology of the scale-free network studied in the case of two opinions. The transition to consensus as a function of noise for the two scale-free networks seems to belong to the same type of transition as in the case of annealed interactions. In contrast, we compare the behavior of the model with a regular network with $k=6$ on a chain (in the previous section we had $k=2$), adding interactions up to the third nearest neighbors. The transition from consensus to a uniform distribution on the regular network differs from the transition of complex networks and annealed case and presents similar behavior as the one reported in previous sections for a chain.

In Fig. 8(a) we show $\langle D \rangle$ vs $\eta$ for the model on the BA (triangles) and Apollonian networks (solid line), compared to the result of annealed interactions (plus signs) and the regular network (circles). The results of the figure represent the average over 20 realizations on systems of $N=124$ agents and 2 opinions. Near the transition, the fluctuations on the regular lattice strongly increase, as opposed to the annealed interaction and to BA and Apollonian networks. This is because the system presents an intermittency near the transition point ($\eta \approx 0.4$). We observe this intermittency of the dynamics in Fig. 8(c), comparing the value of the dominant opinion $a_{o_{max}}$ vs *time*, for the Apollonian and the regular network with $\eta=0.4$. Above the critical noise, there is no consensus and the fraction of agents with dominant opinion is $\langle D \rangle \approx 1/2$. At these values of $\eta$, the response of the system is similar for scale-free and regular networks, as is shown in Fig. 8(d) with $\eta=0.6$.

Above the critical noise there is no way for the agents to achieve global coordination. In this situation, the dynamics is dominated by local interactions, and thus the topology of the system has little effect on this regime. Below the critical noise, global coordination becomes possible. However, the low dimensionality of the regular lattice leads to the intermittent behavior observed in the panel 8(c).

## IV. CONCLUSION

Starting from a model based on interactions with a term of second order, we analyzed its behavior for different topologies: random, regular, and complex ones. Depending on the control parameter, the noise $\eta$, two different absorbing states control the system. Its behavior changes from consensus to a uniform distribution of opinions. Despite the rather simple approach to take into account such simple interactions, a rich variety of results can be reported depending on the dimension of the opinion vector. The results show that an opinion is kept (for systems with more than two opinions) and the parameters need to be adjusted crucially to change the state (hysteresis). This occurs at different dimensions $O$ of the opinion vector, depending on the topology of interactions.

The response of the system to approach consensus has its origin in the model dynamics as opposed to the particular features of the network. An important characteristic of the transition to consensus is the dimension associated with the space of agent interactions. The dynamical response of the opinion model for both scale-free networks is similar to the response observed for annealed interactions and each of these cases represents long-range interactions. In contrast, differences are reported with a regular lattice, which has spatial dimension 1, associated with nearest-neighbor interactions.

As was previously observed for the Sznajd-Weron–Sznajd model of opinion formation, for the general model that we present here, the response of the system in terms of opinion formation is qualitatively the same for a deterministic scale-free network as well as for a random scale free network. This implies a clear advantage for an analytical treatment in a similar way as was done for the Sznajd-Weron–Sznajd model [15].

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